Eigenvalues and methods for finding eigenvalues

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Administrative Stuff

- Second office hour:
  - Wednesday 12:45pm - 1:45pm @LS5229

Eigenvalues and eigenvector

- Definition:
  - Given a matrix A, then the eigenvalue, λ, and eigenvector v are defined as:
  - $Av = \lambda v$
  - $\Rightarrow (A - \lambda I)v = \mathbf{0}$ (here I is the identical matrix).
  - Assume v is not zero vector, then $M = A - \lambda I$ is causing the product equal to zero matrix.
  - M is not invertible, meaning that the determinant of M equal to 0.

Eigenvalues

- For 2 by 2 matrix:
  - Assume that diagonal entry equals to the same negative value:

$$A = \begin{bmatrix} -a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix}$$

  - Then $det(A - \lambda I) = 0$ meaning that:
  - $$det\left(\begin{bmatrix} -a_1 - \lambda & a_2 \\ a_3 & -a_1 - \lambda \end{bmatrix} \right) = \lambda^2 - 2a_1\lambda + (a_1^2 - a_2 a_3)$$
  - $\lambda_{\pm} = \frac{2a_1 \pm \sqrt{4a_1^2 - 4(a_1^2 - a_2 a_3)}}{2}$

Eigenvalues

- Use `eig()` in Matlab to find the analytical solution for 2 by 2 matrix.
- Try 3 by 3 and 4 by 4 matrix.
- We can see that the number of terms for the eigenvalue solutions exploded for even 3 by 3 matrix.
- What is the largest size of matrix can be solved in terms of eigenvalue?

Eigenvalues

- Answer: generally 4 by 4 matrix.
- So how to find eigenvalues in practice for n by n matrix?
- for n by n matrix, calculating the eigenvalues is equivalent as computing the roots of nth order polynomial.
- So how to solve nth-order polynomial numerically?

Eigenvalues

- There are lots of good numerical solvers for polynomials in Matlab.
- Given polynomial: $p_1 x^n + ... + p_n x + p_{n+1} = 0$
- `Roots([p1 p2 p3 ...  pn pn+1])` outputs the solutions for the polynomial.

Eigenvalues

- Iterative algorithm for finding eigenvalues:
  - Power iteration or Von Mises iteration.
  - Given n by n matrix A (assume symmetric and real) and a random 1 byn vector v.
  - Since eigenvectors $(v_1, v_2, ..., v_n)$ form a basis for vector space associated with A.
  - Then we can express v as a linear combination of eigenvectors:
  - $v = a_1 v_1 + a_2 v_2 + ... + a_n v_n$
  - For some coefficient: $a_1, a_2, ..., a_n$

Eigenvalues

- Then (assume eigenvalues: $\lambda_1, \lambda_2, ..., \lambda_n$):
  - $$\begin{aligned} Av &= Aa_1 v_1 + Aa_2 v_2 + ... + Aa_n v_n \\ &= a_1\lambda_1 v_1 + a_2\lambda_2 v_2 + ... + a_n\lambda_n v_n \\ &= a_1\lambda_1 (v_1 + \frac{a_2}{a_1}\frac{\lambda_1}{\lambda_2} v_2 + ... + \frac{a_n}{a_1}\frac{\lambda_n}{\lambda_1} v_n) \end{aligned}$$
  - Assume that λ_1 is the eigenvalue with largest magnitude, then:
  - $\frac{\lambda_2}{\lambda_1} < 1, \frac{\lambda_3}{\lambda_1} < 1, ..., \frac{\lambda_n}{\lambda_1} < 1$
  - What happen if we multiply A for i times?

Eigenvalues

- Then
  - $$\begin{aligned} A^i v &= A^i a_1 v_1 + A^i a_2 v_2 + \ldots + A^i a_n v_n \\ &= A^{i-1} a_1 \lambda_1 v_1 + A^{i-1} a_2 \lambda_2 v_2 + \ldots + A^{i-1} a_n \lambda_n v_n \\ &= A^{i-2} a_1 \lambda_1^2 v_1 + A^{i-2} a_2 \lambda_2^2 v_2 + \ldots + A^{i-2} a_n \lambda_n^2 v_n \\ &= \ldots \\ &= a_1 \lambda_1^i v_1 + a_2 \lambda_2^i v_2 + \ldots + a_n \lambda_n^i v_n \\ &= a_1 \lambda_1^i \left(v_1 + \frac{a_2}{a_1}\left(\frac{\lambda_1}{\lambda_2}\right)^i v_2 + \ldots + \frac{a_n}{a_1}\left(\frac{\lambda_n}{\lambda_1}\right)^i v_n\right) \end{aligned}$$
  - What if i is large?

Eigenvalues

- Then
  - $u_i = A^i v \approx a_1 \lambda_1^i v_1$
  - Now divide by the norm of the vectors and make this a unit vector (normalize for higher and higher power of i):
  - $u_i^* = \frac{u_i}{norm(u_i)} = \frac{a_1 \lambda_1^i v_1}{a_1 \lambda_1^i norm(v_1)} = v_1$
  - Also note that $u_i = A^i v = AA^{i-1} v = Au_{i-1}$
  - For larger and larger i, u_i^* converges to eigenvector corresponding to the largest eigenvalue, i.e. v_1.

Eigenvalues

- Then for large i:
  - We can assume that $u_i^* = u_{i-1}^*$
  - Then, $u_i^* = Au_{i-1}^* = Au_i^*$
  - Then one way to find the largest eigenvalue (λ_1) is:
  - $$\begin{aligned} \mathbf{u_i^{*\prime} A u_i^*} &= \frac{u_i'}{norm(u_i')} A \frac{u_i}{norm(u_i)} \\ &= \frac{a_1 \lambda_1^i v_1'}{a_1 \lambda_1^i norm(v_1')} A \frac{a_1 \lambda_1^i v_1}{a_1 \lambda_1^i norm(v_1)} \\ &= v_1' A v_1 = v_1' \lambda_1 v_1 = \lambda_1 v_1' v_1 = \lambda_\mathbf{1} \end{aligned}$$
  - **Note that u' is the transpose of u.**

Eigenvalues

- Therefore:
  - To find the eigenvalue with largest magnitude:
  - Each iteration calculate $\frac{u_i}{norm(u_i)} = \frac{A^i v}{norm(A^i v)}$
  - For large enough i: $\mathbf{u_i' A u_i} = \lambda_1$